

LONG EXACT SEQUENCES FOR DE RHAM COHOMOLOGY OF DIFFEOLOGICAL SPACES

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ABSTRACT. In this paper we present the notion of de Rham cohomology with compact support for diffeological spaces. Moreover we shall discuss the existence of three long exact sequences. As a concrete example, we show that long exact sequences exist for the de Rham cohomology of diffeological subcartesian spaces.

1. INTRODUCTION

Generally, the de Rham cohomology is a cohomology based on differential forms of a topological smooth manifold. In [7], J.-M. Souriau introduced diffeological spaces as generalization of the notions of topological smooth manifolds. Moreover, In [2], P. Iglesias-Zemmour extended the notions of differential forms and de Rham cohomology groups on diffeological spaces.

In Section 2 we discuss about the Hausdorffness, the compactness, the paracompactness, and the normality of D -topology of a diffeological space. Since the inclusion is not compatible with D -topologies, we need to be cautious in dealing with these notions. But we can show that if a diffeological space is D -paracompact and D -Hausdorff, then it is D -normal. In Section 3 we introduce "diffeological subcartesian spaces" which is a diffeological space locally diffeomorphic to a (not necessarily open) subspace of an Euclidean space. It is shown that every diffeological subcartesian space has a partition of unity subordinate to any D -open cover. In Section 4 we discuss de Rham cohomology (with compact support) in respect to diffeological spaces. It is shown that if there is a D -open cover $\{A, B\}$ of X such that there exists a partition of unity subordinate to it, then we have a Mayer-Vietoris exact sequence of de Rham cohomology groups (see Theorem 4.3):

$$\rightarrow H_{\text{dR}}^p(X) \xrightarrow{j_1^* \oplus j_2^*} H_{\text{dR}}^p(A) \oplus H_{\text{dR}}^p(B) \xrightarrow{i_1^* - i_2^*} H_{\text{dR}}^p(A \cap B) \xrightarrow{\delta} H_{\text{dR}}^{p+1}(X) \rightarrow \cdots$$

On the other hand, we shall see in Section 5 that if X is D -Hausdorff, then there is a Mayer-Vietoris exact sequence of de Rham cohomology groups with compact support (see Theorem 5.3):

$$\rightarrow H_c^p(A \cap B) \xrightarrow{i_1^* \oplus i_2^*} H_c^p(A) \oplus H_c^p(B) \xrightarrow{j_1^* - j_2^*} H_c^p(X) \xrightarrow{\delta} H_c^{p+1}(A \cap B) \rightarrow \cdots$$

In particular, if X is a diffeological subcartesian space, then there exist both types of Mayer-Vietoris exact sequences. In Section 6, we introduce a long exact sequence for pair of diffeological spaces. Let A be a D -compact set of a diffeological subcartesian

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space X . If there exists a D -open set M of X such that A is a deformation retract of M , then we have a long exact sequence (see Theorem 6.2):

$$\rightarrow H_c^p(X \setminus A) \xrightarrow{i_*} H_c^p(X) \xrightarrow{j^*} H_c^p(A) \xrightarrow{\delta} H_c^{p+1}(X \setminus A) \rightarrow \cdots.$$

I would like to thank Kazuhisa Shimakawa, who suggested me the idea of using differential p -forms on a diffeological space with compact support (see Definition 5.1).

2. THE D -TOPOLOGY FOR DIFFEOLOGICAL SPACES

A diffeological space consists of a set X together with a family D of maps from open subsets of Euclidean spaces into X satisfying the following conditions:

Covering: Any constant parametrization $\mathbf{R}^n \rightarrow X$ belongs to D .

Locality: A parametrization $P: U \rightarrow X$ belongs to D if every point u of U has a neighborhood W such that $P|_W: W \rightarrow X$ belongs to D .

Smooth compatibility: If $P: U \rightarrow X$ belongs to D , then so does the composite $P \circ Q: V \rightarrow X$ for any smooth map $Q: V \rightarrow U$ between open subsets of Euclidean spaces.

We call D a diffeology of X , and each member of D a plot of X . A map $f: X \rightarrow Y$ between diffeological spaces is called smooth if for any plot $P: U \rightarrow X$ of X , the composite $f \circ P: U \rightarrow Y$ is a plot of Y . Clearly, the class of diffeological spaces and smooth maps form a category **Diff**.

Theorem 2.1 ([2, 1.60], [6, 2.1]). *The category **Diff** is complete, cocomplete, and cartesian closed.*

Let X be a diffeological space. Let A be a subset of X . We say that A is D -open in X if for any plot $P: U \rightarrow X$ of X , $P^{-1}(A)$ is open in U . A subset A is called D -closed in X if for any plot $P: U \rightarrow X$ of X , $P^{-1}(A)$ is closed in U . We will denote the closure of A by \overline{A} . That is to say, \overline{A} is the smallest D -closed subset containing A .

Lemma 2.2 ([1, 3.17]). *Let A be a D -open set of diffeological space X . Then a subset B of A is D -open in X if and only if it is D -open in A .*

Remark. Let A be a subset of a diffeological space X . Then we can give A two topologies:

- (1) $\tau_1(A)$: the D -topology of the sub-diffeology on A ;
- (2) $\tau_2(A)$: the sub-topology of the D -topology on X .

However, these topologies are not always the same. In general, we can only conclude that $\tau_2(A) \subseteq \tau_1(A)$. Therefore we need to be careful when defining separation axioms and compactness.

Definition 2.3 (D -Hausdorff space). A diffeological space X is D -Hausdorff if and only if for any elements x and y of X , there are D -open neighborhoods U_x and U_y of x and y , respectively, such that $U_x \cap U_y = \emptyset$.

We have the following by the above remark.

Lemma 2.4. *Let A be a subspace of X . If X is D -Hausdorff, then A is also D -Hausdorff.*

Definition 2.5 (*D*-compact space). Let C be a subset of a diffeological space X . We say that C is *D*-compact in X if every covers of C consisting of *D*-open sets of X have a finite cover. If X is *D*-compact in X , then it is called to be *D*-compact.

Then we have the following by Lemma 2.2.

Proposition 2.6. *Let A be a *D*-open set of X . Then a subset C of A is *D*-compact in X if and only if it is *D*-compact in A .*

It is not difficult to prove the following.

Proposition 2.7. *If X is *D*-compact, then every *D*-closed subset of X is also *D*-compact in X .*

Proposition 2.8. *If X is *D*-Hausdorff, then every *D*-compact subset of X is *D*-closed in X .*

We turn to *D*-paracompactness. Let X be a diffeological space. A collection $\{W_\lambda\}_{\lambda \in \Lambda}$ of subsets of X is called locally finite if each $x \in X$ has an *D*-open neighborhood whose intersection with W_λ is non-empty only for finitely many λ .

Lemma 2.9. *Let $\{W_\lambda\}_{\lambda \in \Lambda}$ be a collection of subsets of X . If $\{W_\lambda\}_{\lambda \in \Lambda}$ is locally finite, then $\cup_{\lambda \in \Lambda} \overline{W_\lambda} = \overline{\cup_{\lambda \in \Lambda} W_\lambda}$ holds.*

Proof. It is clear that $\cup_{\lambda \in \Lambda} \overline{W_\lambda} \subset \overline{\cup_{\lambda \in \Lambda} W_\lambda}$ holds. Conversely, let $x \notin \cup_{\lambda \in \Lambda} \overline{W_\lambda}$. Then for any $\lambda \in \Lambda$, there exists *D*-open neighborhood $U_\lambda(x)$ of x such that $U_\lambda(x) \cap W_\lambda = \emptyset$. Since $\{W_\lambda\}_{\lambda \in \Lambda}$ is locally finite, there exist a *D*-open neighborhood V of x and finitely many $\lambda_i \in \Lambda$ ($1 \leq i \leq m$) such that $V \cap W_{\lambda_i} \neq \emptyset$. Let $U = (\cap_{1 \leq i \leq m} U_{\lambda_i}(x)) \cap V$. Then U is *D*-open neighborhood of x . Since for any $\lambda \in \Lambda$, $W_\lambda \cap U = \emptyset$, we have $(\cup_{\lambda \in \Lambda} W_\lambda) \cap U = \emptyset$. Therefore $x \notin \overline{\cup_{\lambda \in \Lambda} W_\lambda}$. \square

Let $\{V_\alpha\}_{\alpha \in I}$ and $\{U_\beta\}_{\beta \in J}$ be two covers of X . We say that $\{V_\alpha\}_{\alpha \in I}$ is a refinement of $\{U_\beta\}_{\beta \in J}$ if for any $\alpha \in I$, there exists $\beta \in J$ such that $V_\alpha \subset U_\beta$.

Definition 2.10 (*D*-paracompact space). We say that a subset A of a diffeological space X is *D*-paracompact in X if every cover of A consisting of *D*-open sets of X has a locally finite refinement consisting of *D*-open sets of X . If X is *D*-paracompact in X , then we call it *D*-paracompact.

Proposition 2.11. *Let A be a *D*-closed subset of X . If X is *D*-paracompact, then A is *D*-paracompact in X .*

Proof. Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be a cover of A consisting of *D*-open sets of X . Then $\mathcal{U} = \{U_\lambda\}_{\lambda \in \Lambda} \cup \{X \setminus A\}$ is *D*-open cover of X . Since X is *D*-paracompact, there exists a locally finite refinement $\{V_\alpha\}_{\alpha \in I}$ of \mathcal{U} . Let $I' = \{\alpha \in I \mid V_\alpha \cap A \neq \emptyset\}$. Then $\{V_\alpha\}_{\alpha \in I'}$ is a locally finite refinement of $\{U_\lambda\}_{\lambda \in \Lambda}$. \square

Definition 2.12 (*D*-normal space). We say that a diffeological space X is *D*-normal if for any *D*-closed sets A and B of X such that $A \cap B = \emptyset$, there exist *D*-open sets U_A and U_B of X such that $A \subset U_A$, $B \subset U_B$ and $U_A \cap U_B = \emptyset$.

From the definition, it is clear that we have the following.

Proposition 2.13. *A diffeological space X is *D*-normal if and only if for any *D*-closed set A and *D*-open set B of X such that $A \subset B$, there exists a *D*-open set U_A of X such that $A \subset U_A \subset \overline{U_A} \subset B$.*

Theorem 2.14. *If X is D -Hausdorff and D -paracompact, then it is D -normal.*

Proof. Let x be an element of X . Let F be a D -closed set of X such that $x \notin F$ and let y be an element of F . Since X is D -Hausdorff, there exists a D -open neighborhood U_x and U_y of x and y , respectively, such that $U_x \cap U_y = \emptyset$. Then $\mathcal{U} = \{U_y | y \in F\} \cup \{X \setminus F\}$ is a D -open cover of X . Since X is D -paracompact, there exists a locally finite refinement $\{W_\lambda\}_{\lambda \in \Lambda}$ of \mathcal{U} . Thus there are a D -open neighborhood V of x and finitely many λ_i ($1 \leq i \leq m$) such that $W_{\lambda_i} \cap V \neq \emptyset$. Let $I = \{\lambda_i | x \notin W_{\lambda_i}, 1 \leq i \leq m\}$ and $V_0 = V \setminus \bigcup_{\lambda_i \in I} \overline{W_{\lambda_i}}$. Since $\bigcup_{\lambda_i \in I} \overline{W_{\lambda_i}} = \overline{\bigcup_{\lambda_i \in I} W_{\lambda_i}}$ holds, V_0 is a D -open neighborhood of x . Let $J = \{\lambda \in \Lambda | W_\lambda \cap F \neq \emptyset\}$ and $W = \bigcup_{\lambda \in J} W_\lambda$. Then W is a D -open set of X such that $F \subset W$. Then we have $V_0 \cap W = \emptyset$.

Next, let A and B be D -closed sets of X such that $A \cap B = \emptyset$. By the above condition, for any $a \in A$, there exist a D -open neighborhood U_a of a and D -open set $W(a)$ of X such that $B \subset W(a)$ and $U_a \cap W(a) = \emptyset$. Let $\mathcal{U}' = \{U_a | a \in A\} \cup \{X \setminus A\}$. Then \mathcal{U}' is a D -open cover of X . Since X is D -paracompact, there exists a locally finite refinement $\{L_\lambda\}_{\lambda \in \Lambda}$ of \mathcal{U}' . For any $b \in B$, there exist a D -open neighborhood M_b of b and finitely many λ_i ($1 \leq i \leq m$) such that $M_b \cap L_{\lambda_i} \neq \emptyset$. Let $I = \{\lambda_i | a \notin L_{\lambda_i}, 1 \leq i \leq m\}$ and let $M = \bigcup_{b \in B} \tilde{M}_b$, where $\tilde{M}_b = M_b \setminus \bigcup_{\lambda_i \in I} \overline{L_{\lambda_i}}$ is a D -open neighborhood of b . Let $J = \{\lambda \in \Lambda | L_\lambda \cap A \neq \emptyset\}$ and let $L = \bigcup_{\lambda \in J} L_\lambda \supset A$. Then we have $L \cap M = \emptyset$. \square

3. DIFFEOLOGICAL SUBCARTESIAN SPACES

In this section we present the notion of diffeological subcartesian spaces. Moreover we prove that every diffeological subcartesian space has a partition of unity subordinate to arbitrary D -open cover.

Definition 3.1 (diffeological subcartesian space). We say that a diffeological space X is a diffeological subcartesian space if the following conditions are satisfied.

- (1) X is D -Hausdorff and D -paracompact.
- (2) For each x in X , there exists a diffeomorphism $\varphi_x: U \rightarrow U'$, called a chart at x , from a D -open neighborhood U of x to a subspace U' of an Euclidean space \mathbf{R}^{n_x} , where $n_x \geq 0$ and U' need not be open in \mathbf{R}^{n_x} .

It is clear that a diffeological subcartesian space is D -normal by Theorem 2.14.

Lemma 3.2 (Shrink Lemma). *If a diffeological space X is D -Hausdorff and D -paracompact, then for every D -open cover $\{U_\lambda\}_{\lambda \in \Lambda}$ of X , there exists a locally finite D -open cover $\{W_\lambda\}_{\lambda \in \Lambda}$ such that $\overline{W_\lambda} \subset U_\lambda$ holds for any $\lambda \in \Lambda$.*

Proof. Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be a D -open cover of X . Since $\bigcup_{\lambda \in \Lambda} U_\lambda = X$ holds, we have

$$\bigcap_{\lambda \in \Lambda} (X \setminus U_\lambda) = \bigcap_{\lambda \neq \lambda' \in \Lambda} (X \setminus U_{\lambda'}) \cap (X \setminus U_\lambda) = \emptyset.$$

Since X is D -normal by Theorem 2.14, there exist two D -open sets S_λ and W of X such that

$$\bigcap_{\lambda \neq \lambda' \in \Lambda} (X \setminus U_{\lambda'}) \subset S_\lambda, \quad X \setminus U_\lambda \subset W \quad \text{and} \quad S_\lambda \cap W = \emptyset.$$

Thus we have $\overline{S_\lambda} \subset U_\lambda$ and $S_\lambda \cup (\bigcup_{\lambda \neq \lambda' \in \Lambda} X \setminus U_{\lambda'}) = X$ since $S_\lambda \subset X \setminus W \subset U_\lambda$ holds. Let $\mathcal{U} = \{S_\lambda | \lambda \in \Lambda\}$. Then \mathcal{U} is a D -open cover of X . Since X is D -paracompact, there exists a locally finite refinement $\{W_j\}_{j \in J}$ of \mathcal{U} . For any $\lambda \in \Lambda$,

let $J_\lambda = \{j_\lambda \in J \mid W_{j_\lambda} \subset S_\lambda\}$ and let $V_\lambda = \cup_{j_\lambda \in J_\lambda} W_{j_\lambda}$. Then we have

$$\overline{V}_\lambda = \overline{\cup_{j_\lambda \in J_\lambda} W_{j_\lambda}} \subset \overline{S_\lambda} \subset U_\lambda.$$

Then $\{V_\lambda \mid \lambda \in \Lambda\}$ is a locally finite D -open cover of X . \square

Let X be a diffeological space. If $f: X \rightarrow \mathbf{R}$ is a real-valued smooth map on X , the support of f , denoted by $\text{supp} f$, is the closure of the set of points where f is nonzero:

$$\text{supp} f = \overline{\{p \in X; f(p) \neq 0\}}.$$

Let $\mathbf{G} = \{A_\lambda\}_{\lambda \in \Lambda}$ be an arbitrary D -open cover of X . We say that a collection $\{\phi_\lambda: X \rightarrow \mathbf{R}\}_{\lambda \in \Lambda}$ is a partition of unity subordinate to \mathbf{G} if the following conditions are satisfied:

- (1) $0 \leq \phi_\lambda(x) \leq 1$ for all $\lambda \in \Lambda$ and all $x \in X$,
- (2) $\text{supp} \phi_\lambda \subset A_\lambda$,
- (3) the set $\{\text{supp} \phi_\lambda\}_{\lambda \in \Lambda}$ of supports is locally finite, and
- (4) $\sum_{\lambda \in \Lambda} \phi_\lambda(x) = 1$ for all $x \in X$.

Theorem 3.3. *Let X be a diffeological subcartesian space. Then for any D -open cover \mathbf{U} of X , there exists a partition of unity subordinate to \mathbf{U} .*

Proof. Let $\mathbf{U} = \{U_\lambda\}_{\lambda \in \Lambda}$ be a D -open cover of X . Without loss of generality we may assume that the elements of \mathbf{U} are chart domains. By Shrink Lemma, there exist locally finite D -open covers $\{V_\lambda\}$ and $\{W_\lambda\}$ of X such that

$$\overline{V}_\lambda \subset W_\lambda \subset U_\lambda.$$

Let $\varphi_\lambda: U_\lambda \rightarrow U'_\lambda$ be a chart for each $\lambda \in \Lambda$ and U'_λ is the subset of \mathbf{R}^{n_λ} . Then there are a closed subset \tilde{V}_λ and an open subset \tilde{W}_λ in \mathbf{R}^{n_λ} such that $\varphi_\lambda^{-1}(\tilde{V}_\lambda) = \overline{V}_\lambda$ and $\varphi_\lambda^{-1}(\tilde{W}_\lambda) = W_\lambda$, respectively. By [4, 2.19], there exists a smooth function $g_\lambda: \mathbf{R}^{n_\lambda} \rightarrow \mathbf{R}$ such that $\text{supp} g_\lambda \subset \tilde{W}_\lambda$, $g_\lambda|_{\tilde{V}_\lambda} \equiv 1$ and $g_\lambda|_{\mathbf{R}^{n_\lambda} \setminus \tilde{W}_\lambda} \equiv 0$. We define $f_\lambda: X \rightarrow \mathbf{R}$ by

$$f_\lambda(x) = \begin{cases} g_\lambda \circ \varphi_\lambda(x) & x \in U_\lambda \\ 0 & x \in X \setminus U_\lambda. \end{cases}$$

Define new function $\phi_\lambda: X \rightarrow \mathbf{R}$ by

$$\phi_\lambda(x) = \frac{f_\lambda(x)}{\sum_{\lambda' \in \Lambda} f_{\lambda'}(x)}.$$

Then $\{\phi_\lambda: X \rightarrow \mathbf{R} \mid \lambda \in \Lambda\}$ is a partition of unity subordinate to \mathbf{U} . \square

Corollary 3.4. *Let A be a D -closed subset of a diffeological subcartesian space X . Let U_A be a D -open subset of X containing A . Then there exists a function $\varphi: X \rightarrow \mathbf{R}$ such that $\text{supp} \varphi \subset U_A$ and $\varphi \equiv 1$ on A .*

Proof. Since $\{U_A, X \setminus A\}$ is a D -open cover of X , there exists a partition of unity subordinate $\{\varphi_{U_A}, \varphi_{X \setminus A}\}$ to $\{U_A, X \setminus A\}$ by Theorem 3.3. Since $\varphi_{X \setminus A} \equiv 0$ on A , the function φ_{U_A} has the required properties. \square

4. DE RHAM COHOMOLOGY OF DIFFEOLOGICAL SPACES

In this section we shall show that there exists a Mayer-Vietoris exact sequence with respect to de Rham cohomology of diffeological spaces.

We first recall from [2] the notion of differential forms on a diffeological space. A covariant antisymmetric p -tensor of \mathbf{R}^n [2, 6.11] is called a linear p -form of \mathbf{R}^n . The vector space of linear p -forms of \mathbf{R}^n is denoted by $\Lambda^p(\mathbf{R}^n)$. Let U be an open set of \mathbf{R}^n . Let d be a linear map from $C^\infty(U, \Lambda^p(\mathbf{R}^n))$ to $C^\infty(U, \Lambda^{p+1}(\mathbf{R}^n))$ defined by [2, 6.24]. Then we have $d \circ d = 0$.

Definition 4.1 ([2, 6.28]). Let X be a diffeological space. We say that α is a differential p -form on X if the following two conditions are fulfilled

- (1) For all integers n , for all n -plots $P: U \rightarrow X$, we have

$$\alpha(P) \in C^\infty(U, \Lambda^k(\mathbf{R}^n)).$$

- (2) For all open sets V of \mathbf{R}^m , $m \geq 0$, for all smooth parametrizations $F: V \rightarrow U$, we have

$$\alpha(P \circ F) = F^*(\alpha(P)),$$

where $F^*(\alpha(P))$ (cf. [2, 6.22]) is defined by

$$F^*(\alpha(P))(v)(x_1) \cdots (x_p) = \alpha(P)(F(v))(D(F)(v)(x_1)) \cdots (D(F)(v)(x_p))$$

for all $v \in V$ and for all k -vectors $x_1, \dots, x_p \in \mathbf{R}^m$.

The condition $\alpha(P \circ F) = F^*(\alpha(P))$ is called the smooth compatibility condition. The set of differential p -forms on X is clearly a real vector space, and will be denoted by $\Omega^p(X)$. We define a linear map d from $\Omega^p(X)$ to $\Omega^{p+1}(X)$ by

$$(d\alpha)(P) = d(\alpha(P))$$

for any $\alpha \in \Omega^p(X)$ and any plot $P: U \rightarrow X$ of X . Clearly, $d \circ d = 0$ holds. Therefore we have a cochain complex $\{\Omega^p(X), d\}$, called the de Rham complex. We define

$$\begin{aligned} Z^p(X) &= \text{Ker}[d: \Omega^p(X) \rightarrow \Omega^{p+1}(X)] \text{ and} \\ B^p(X) &= \text{Im}[d: \Omega^{p-1}(X) \rightarrow \Omega^p(X)]. \end{aligned}$$

Since $B^p(X)$ is a linear subspace of $Z^p(X)$, we can define the p -th de Rham cohomology group of X to be the quotient vector space

$$H_{dR}^p(X) = Z^p(X)/B^p(X).$$

Let $f: X \rightarrow Y$ be a smooth map between diffeological spaces. We define

$$f^*: \Omega^p(Y) \rightarrow \Omega^p(X)$$

by $f^*(\alpha)(P) = \alpha(f \circ P)$ for any $\alpha \in \Omega^p(Y)$ and any plot P of X . Then we have $f^*(d\alpha) = d(f^*\alpha)$ [2, 6.25]. Thus f induces a homomorphism $f^*: H_{dR}^p(Y) \rightarrow H_{dR}^p(X)$. Let A be a D -open set of X . For any $\alpha \in \Omega^p(X)$, we define $\alpha|_A = i_A^*(\alpha) \in \Omega^p(A)$, where $i_A: A \rightarrow X$ is the inclusion map.

Proposition 4.2. *Let X be a diffeological space. Let $\{X_\lambda\}$ be a collection of subspaces of X such that $X = \coprod_{\lambda \in \Lambda} X_\lambda$. Then $H_{dR}^p(X)$ and $\prod_{\lambda \in \Lambda} H_{dR}^p(X_\lambda)$ are isomorphic for each p .*

Proof. It is clear that for each $\lambda \in \Lambda$, X_λ is D -open in X by the definition of coproduct spaces. We define a homomorphism

$$\prod i_\lambda^*: \Omega^p(X) \rightarrow \prod_{\lambda \in \Lambda} \Omega^p(X_\lambda)$$

by $\prod i_\lambda^*(\omega) = (i_\lambda^*(\omega))_{\lambda \in \Lambda}$ for any $\omega \in \Omega^p(X)$, where $i_\lambda^*: \Omega^p(X) \rightarrow \Omega^p(X_\lambda)$ is the cochain map induced by the inclusion $i_\lambda: X_\lambda \rightarrow X$. Let ω be an element of $\text{Ker}(\prod i_\lambda^*)$. Since $(\prod i_\lambda^*)(\omega) = ((i_\lambda^*(\omega))_{\lambda \in \Lambda}) = (\omega|_{X_\lambda})_{\lambda \in \Lambda} = 0$, we have $\omega = 0$. Let $(\tau_\lambda)_{\lambda \in \Lambda}$ be an element of $\prod_{\lambda \in \Lambda} \Omega^p(X_\lambda)$. We define $\tau \in \Omega^p(X)$ by for any $\lambda \in \Lambda$, $\tau|_{X_\lambda} = \tau_\lambda$. Then we have $(\prod i_\lambda^*)(\tau) = (\tau_\lambda)_{\lambda \in \Lambda}$ since $X_\lambda \cap X_{\lambda'}$ is empty for each $\lambda \neq \lambda'$. \square

Let A and B be two D -open sets of X such that $X = A \cup B$ holds. Then we have a diagram:

$$(1) \quad \begin{array}{ccc} A \cap B & \xrightarrow{i_1} & A \\ i_2 \downarrow & & j_1 \downarrow \\ B & \xrightarrow{j_2} & X = A \cup B \end{array}$$

consisting of inclusions. Now, consider the following sequence:

$$(2) \quad 0 \rightarrow \Omega^p(X) \xrightarrow{j_1^* \oplus j_2^*} \Omega^p(A) \oplus \Omega^p(B) \xrightarrow{i_1^* - i_2^*} \Omega^p(A \cap B) \rightarrow 0,$$

where

$$(j_1^* \oplus j_2^*)(\omega) = (j_1^*(\omega), j_2^*(\omega)), \quad (i_1^* - i_2^*)(\omega) = i_1^*(\omega) - i_2^*(\omega).$$

Then we have the following.

Theorem 4.3 (Mayer-Vietoris exact sequence). *Let X be a diffeological space. Let $\{A, B\}$ be a D -open cover of X . If there exists a partition of unity $\varphi_i: X \rightarrow \mathbf{R}$ ($i = A, B$) subordinate to $\{A, B\}$, then we have a long exact sequence:*

$$\rightarrow H_{\text{dR}}^p(X) \xrightarrow{j_1^* \oplus j_2^*} H_{\text{dR}}^p(A) \oplus H_{\text{dR}}^p(B) \xrightarrow{i_1^* - i_2^*} H_{\text{dR}}^p(A \cap B) \xrightarrow{\delta} H_{\text{dR}}^{p+1}(X) \rightarrow \dots$$

Proof. To see existence of the Mayer-Vietoris exact sequence, it suffices to show that the sequence (2) is exact for each p . We shall show that $j_1^* \oplus j_2^*$ is injective. Let α be an element of $\text{Ker}(j_1^* \oplus j_2^*)$. Since $\alpha|_A = 0 = \alpha|_B$, we have $\alpha = 0$. Let $(j_1^* \oplus j_2^*)(\omega)$ be an element of $\text{Im}(j_1^* \oplus j_2^*)$. Since $i_1^* j_1^*(\alpha) = \alpha|_{A \cap B} = i_2^* j_2^*(\alpha)$, we have

$$(i_1^* - i_2^*) \circ (j_1^* \oplus j_2^*)(\omega) = i_1^* j_1^*(\omega) - i_2^* j_2^*(\omega) = 0.$$

Thus $\text{Im}(j_1^* \oplus j_2^*) \subset \text{Ker}(i_1^* - i_2^*)$. Let (α, β) be an element of $\text{Ker}(i_1^* - i_2^*)$. We define $\omega \in \Omega^p(X)$ by

$$\omega = \begin{cases} \alpha & \text{on } A \\ \beta & \text{on } B, \end{cases}$$

that is, for any plot $P: U \rightarrow X$ of X , $\omega(P) = \alpha(P|_{P^{-1}(A)})$ on $P^{-1}(A)$ and $\omega(P) = \beta(P|_{P^{-1}(B)})$ on $P^{-1}(B)$. Then ω is well-defined since $\alpha|_{A \cap B} = \beta|_{A \cap B}$ holds. Clearly, we have $(j_1^* \oplus j_2^*)(\omega) = (\alpha, \beta)$. Thus $\text{Ker}(i_1^* - i_2^*) \subset \text{Im}(j_1^* \oplus j_2^*)$. We

shall show that $(i_1^* - i_2^*)$ is surjective. Let σ be an element of $\Omega^p(A \cap B)$. Since $\text{supp}\varphi_B \cap A \subset A \cap B$, we can define $\eta_A \in \Omega^p(A)$ by

$$\eta_A = \begin{cases} \varphi_B \times \sigma & \text{on } A \cap B \\ 0 & \text{on } A \setminus \text{supp}\varphi_B, \end{cases}$$

that is, for any plot $P: V \rightarrow A$ of A , $\eta_A(P) = (\varphi_B \circ P|_{P^{-1}(A \cap B)}) \times \sigma(P|_{P^{-1}(A \cap B)})$ on $P^{-1}(A \cap B)$ and $\eta_A(P) = 0$ on $P^{-1}(A \setminus \text{supp}\varphi_B)$. Then η_A satisfies the smooth compatibility condition $F^*(\eta_A(P)) = \eta_A(P \circ F)$ for every smooth map F from an open set of Euclidean spaces to the domain of P . Similarly, we define $\eta_B \in \Omega^p(B)$ by

$$\eta_B = \begin{cases} -\varphi_A \times \sigma & \text{on } A \cap B \\ 0 & \text{on } A \setminus \text{supp}\varphi_A. \end{cases}$$

Then we have $(i_1^* - i_2^*)(\eta_A, \eta_B) = \varphi_B \times \sigma + \varphi_A \times \sigma = (\varphi_B + \varphi_A) \times \sigma = \sigma$. Therefore $(i_1^* - i_2^*)$ is surjective. \square

5. DE RHAM COHOMOLOGY WITH COMPACT SUPPORT

In this section we define the de Rham cohomology of diffeological spaces with compact support. We shall show that there exists a Mayer-Vietoris exact sequence for de Rham cohomology with compact support.

Definition 5.1. Let X be a diffeological space. Let α be a differential p -form on X . An element x in X is a support element of α if and only if for any plot $P: U \rightarrow X$ of X and any $r \in U$ such that $P(r) = x$, $\alpha(P)(r)$ is nonzero. We call the closure of the set of support elements the support of α , and it will be denoted by $\text{supp}\alpha$:

$$\text{supp}\alpha = \overline{\{x \in X \mid \forall P: U \rightarrow X, \forall r \in U \text{ s.t. } P(r) = x, \alpha(P)(r) \neq 0\}}.$$

We say that α is a compactly supported p -form on X if the support of α is D -compact in X . The set of compactly supported p -forms of X is denoted by $\Omega_c^p(X)$.

It is clear that $\{\Omega_c^p(X), d\}$ is the subcomplex of the de Rham complex $\{\Omega^p(X), d\}$. We define the p -th de Rham cohomology group of X with compact support to be the quotient space

$$H_c^p(X) = Z_c^p(X) / B_c^p(X),$$

where,

$$\begin{aligned} Z_c^p(X) &= \text{Ker}[d: \Omega_c^p(X) \rightarrow \Omega_c^{p+1}(X)] \text{ and} \\ B_c^p(X) &= \text{Im}[d: \Omega_c^{p-1}(X) \rightarrow \Omega_c^p(X)]. \end{aligned}$$

We say that a smooth map $f: X \rightarrow Y$ is a proper map if for any D -compact set C in Y , the inverse image $f^{-1}(C)$ is D -compact in X . Now for any element $\alpha \in \Omega_c^p(Y)$, $f^*(\alpha)$ is an element of $\Omega_c^p(X)$ since $f^{-1}(\text{supp}\alpha)$ is D -compact in X . Therefore f induces a homomorphism $f^*: H_c^p(Y) \rightarrow H_c^p(X)$. If X is D -compact, then we have $H_c^p(X) = H_{\text{dR}}^p(X)$ since $\Omega_c^p(X) = \Omega^p(X)$ holds.

Let X be a D -Hausdorff space. Let A be a D -open set of X . If a subset C of A is D -compact in A , then it is D -compact in X by Proposition 2.6. Hence the inclusion $i: A \rightarrow X$ induces a map

$$i_*: \Omega_c^p(A) \rightarrow \Omega_c^p(X)$$

defined by for any $\alpha \in \Omega_c^p(A)$,

$$i_*(\alpha) = \begin{cases} \alpha & \text{on } A \\ 0 & \text{on } X \setminus \text{supp}\alpha, \end{cases}$$

that is, for any plot $P: U \rightarrow X$ of X , $i_*(\alpha)(P) = \alpha(P|_{P^{-1}(A)})$ on $P^{-1}(A)$ and $i_*(\alpha)(P) = 0$ on $P^{-1}(X \setminus \text{supp}\alpha)$. Then it is clear that i_* is injective and $i^* \circ i_*(\alpha) = \alpha$ holds. Moreover we get a homomorphism $i_*: H_c^p(A) \rightarrow H_c^p(X)$.

Proposition 5.2. *Let X be a D -Hausdorff space. Let $\{X_\lambda\}$ be a collection of subspaces of X such that X can be written as a coproduct $X = \coprod_{\lambda \in \Lambda} X_\lambda$. Then $\oplus_{\lambda \in \Lambda} H_c^p(X_\lambda)$ and $H_c^p(X)$ are isomorphic for each p .*

Proof. Let $\sum i_{\lambda*}: \oplus_{\lambda \in \Lambda} \Omega_c^p(X_\lambda) \rightarrow \Omega_c^p(X)$ be the map defined by

$$\left(\sum i_{\lambda*} \right) ((\omega_\lambda)_{\lambda \in \Lambda}) = \sum_{\lambda \in \Lambda} i_{\lambda*}(\omega_\lambda),$$

where $i_{\lambda*}: \Omega_c^p(X_\lambda) \rightarrow \Omega_c^p(X)$ is the chain map induced by the inclusion i_λ . Let $(\omega_\lambda)_{\lambda \in \Lambda}$ be an element of $\text{Ker}(\sum i_{\lambda*})$. For each $\lambda \in \Lambda$ and any plot $P: U \rightarrow X_\lambda$ of X_λ , it is also a plot of X . Then we have

$$\left(\sum i_{\lambda*} \right) ((\omega_\lambda)_{\lambda \in \Lambda})(P) = \sum_{\lambda \in \Lambda} i_{\lambda*}(\omega_\lambda)(P) = \omega_\lambda(P) = 0.$$

Thus $\sum i_{\lambda*}$ is injective since $\omega_\lambda = 0$ for each $\lambda \in \Lambda$. Next we shall show that $\sum i_{\lambda*}$ is surjective. Let τ be an element of $\Omega_c^p(X)$. For each $\lambda \in \Lambda$, X_λ is D -open in X by the properties of coproduct diffeology. Since $\text{supp}\tau$ is D -compact in X and $\{X_\lambda\}_{\lambda \in \Lambda}$ is a cover of $\text{supp}\tau$, there exists a finite cover $\{X_{\lambda_i}\}_{1 \leq i \leq m}$ of $\text{supp}\tau$. We define $(\tau_j)_{j \in \Lambda} \in \oplus_{\lambda \in \Lambda} \Omega_c^p(X_j)$ by

$$(\tau_j)_{j \in \Lambda} = \begin{cases} \tau|_{X_{\lambda_i}} & j = \lambda_i \\ 0 & j \neq \lambda_i. \end{cases}$$

Then $(\tau_j)_{j \in \Lambda}$ is well-defined since $X_\lambda \cap X_{\lambda'}$ is empty for each λ and λ' in Λ such that $\lambda' \neq \lambda$. Clearly, we have $(\sum i_{\lambda*})((\tau_j)_{j \in \Lambda}) = \tau$. \square

Theorem 5.3 (Mayer-Vietoris exact sequence). *Let X be a D -Hausdorff space. Let $\{A, B\}$ be a D -open cover of X . If there exists a partition of unity $\varphi_i: X \rightarrow \mathbf{R}$ ($i = A, B$) subordinate to $\{A, B\}$, then we have a long exact sequence:*

$$\rightarrow H_c^p(A \cap B) \xrightarrow{i_{1*} \oplus i_{2*}} H_c^p(A) \oplus H_c^p(B) \xrightarrow{j_{1*} - j_{2*}} H_c^p(X) \xrightarrow{\delta} H_c^{p+1}(A \cap B) \rightarrow \dots$$

Proof. To see existence of the Mayer-Vietoris exact sequence, it suffices to show that the sequence

$$0 \rightarrow \Omega_c^p(A \cap B) \xrightarrow{i_{1*} \oplus i_{2*}} \Omega_c^p(A) \oplus \Omega_c^p(B) \xrightarrow{j_{1*} - j_{2*}} \Omega_c^p(X) \rightarrow 0$$

is exact for each p . It is not difficult to prove that $i_{1*} \oplus i_{2*}$ is injective and that $\text{Im}(i_{1*} \oplus i_{2*})$ is a linear subspace of $\text{Ker}(j_{1*} - j_{2*})$. Let (α, β) be an element of $\text{Ker}(j_{1*} - j_{2*})$. Since $j_{1*}(\alpha) = j_{2*}(\beta)$ holds, $\alpha|_{A \cap B} = \beta|_{A \cap B}$ and $\alpha = \beta = 0$ on $X \setminus (\text{supp}\alpha \cap \text{supp}\beta)$. Therefore we have $(i_{1*} \oplus i_{2*})(\alpha|_{A \cap B}) = (\alpha, \beta)$. We shall show that $(j_{1*} - j_{2*})$ is surjective. Let ω be an element of $\Omega_c^p(X)$. Then

$j_1^*(\varphi_A \times \omega) \in \Omega_c^p(A)$ and $j_2^*(-\varphi_B \times \omega) \in \Omega_c^p(B)$. Since $\text{supp}(\varphi_A \times \omega) \subset A$ and $\text{supp}(\varphi_B \times \omega) \subset B$, we have

$$\begin{aligned} (j_{1*} - j_{2*})(j_1^*(\varphi_A \times \omega), j_2^*(-\varphi_B \times \omega)) &= j_{1*}j_1^*(\varphi_A \times \omega) - j_{2*}j_2^*(-\varphi_B \times \omega) \\ &= \varphi_A \times \omega + \varphi_B \times \omega \\ &= (\varphi_A + \varphi_B) \times \omega \\ &= \omega. \end{aligned}$$

□

6. LONG EXACT SEQUENCE FOR PAIR OF DIFFEOLOGICAL SPACES

In this section we shall prove Theorem 6.2. Let $f_0, f_1: X \rightarrow Y$ be two smooth maps between diffeological spaces. We say that f_0 and f_1 are homotopic if there exists a homotopy $F: X \times \mathbf{R} \rightarrow Y$ satisfying $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$. If f_0 and f_1 are homotopic then $f_0^* = f_1^*: H_{\text{dR}}^p(Y) \rightarrow H_{\text{dR}}^p(X)$ by [2, 6.88].

Let A be a subspace of a diffeological space X . If there exists a retraction $\gamma: X \rightarrow A$ such that $\gamma|_A = 1_A$, then A is called a retract of X . Moreover we say that A is a deformation retract of X if γ and the identity map $1_X: X \rightarrow X$ are homotopic.

Proposition 6.1. *Let A and C be subsets of a diffeological space X such that $C \subset A$. If there exists a D -open set V of X such that A is a retract of V , then C is D -compact in A if and only if C is D -compact in X .*

Proof. If C is D -compact in A , then it is D -compact in X since all D -open sets of X are D -open in A . Conversely, let C be a D -compact set of X . Let $\{U_\lambda | \lambda \in \Lambda\}$ be a cover of C , where U_λ is D -open in A for each $\lambda \in \Lambda$. Then we have

$$C \subset \cup_{\lambda \in \Lambda} U_\lambda = \cup_{\lambda \in \Lambda} (\gamma^{-1}(U_\lambda) \cap A) \subset \cup_{\lambda \in \Lambda} \gamma^{-1}(U_\lambda),$$

where $\gamma: V \rightarrow A$ is a retraction. Since $\gamma^{-1}(U_\lambda)$ is D -open in V , it is D -open in X by Lemma 2.2. Since C is D -compact in X , $C \subset \cup_{1 \leq i \leq m} \gamma^{-1}(U_{\lambda_i})$ holds. Then we have

$$C \subset \cup_{1 \leq i \leq m} (\gamma^{-1}(U_{\lambda_i}) \cap A) = \cup_{1 \leq i \leq m} U_{\lambda_i}.$$

Therefore C is D -compact in A . □

We will prove the following theorem.

Theorem 6.2. *Let A be a D -compact set in a diffeological subcartesian space X . If there exists a D -open set M of X such that A is a deformation retract of M , then we have a long exact sequence:*

$$\rightarrow H_c^p(X \setminus A) \xrightarrow{i_*} H_c^p(X) \xrightarrow{j^*} H_c^p(A) \xrightarrow{\delta} H_c^{p+1}(X \setminus A) \xrightarrow{i_*} \dots,$$

where $i: X \setminus A \rightarrow X$ and $j: A \rightarrow X$ are inclusions.

We prove this by using the argument similar to that of [3, Proposition 13.11]. The map j coincides with the composite of the inclusions:

$$A \xrightarrow{k_1} M \xrightarrow{k_2} X.$$

Since A is D -compact by Proposition 6.1 and it is a deformation retract of M , a map $\gamma^*: H_c^p(A) = H_{\text{dR}}^p(A) \rightarrow H_{\text{dR}}^p(M)$ is an isomorphism, where $\gamma: M \rightarrow A$ is a retraction. Moreover, there exists a D -open set U_A of X such that

$$A \subset U_A \subset \overline{U_A} \subset M$$

since X is D -normal. Therefore there exists a function $\varphi: X \rightarrow \mathbf{R}$ such that $\text{supp} \varphi \subset M$ and $\varphi \equiv 1$ on \overline{U}_A by Corollary 3.4. Then we have the following lemma.

Lemma 6.3. (1) $j^*: \Omega_c^p(X) \rightarrow \Omega_c^p(A)$ is surjective.
 (2) For any ω in $Z_c^p(A)$, there exists τ in $\Omega_c^p(X)$ such that $j^*(\tau) = \omega$ and $d\tau|_{U_A}$ is zero.
 (3) For any ω in $\Omega_c^p(X)$ such that $\text{supp}(d\tau) \cap A$ is empty and $j^*(\omega)$ is zero, there exists σ in $\Omega_c^{p-1}(X)$ such that $(\omega - d\sigma)|_{U_A}$ is zero.

Proof. We shall show the condition (2). Let ω be an element of $Z_c^p(A)$. Since $\varphi \times \gamma^*(\omega) \in \Omega_c^p(M)$, $\tau = k_{2*}(\varphi \times \gamma^*(\omega))$ is an element of $\Omega_c^p(X)$. Then for any plot $P: U \rightarrow A$ of A , we have

$$\begin{aligned} j^*(\tau)(P) &= j^*k_{2*}(\varphi \times \gamma^*(\omega))(P) \\ &= (\varphi \times \gamma^*(\omega))(P) \\ &= \varphi(P) \times \omega(\gamma \circ P). \end{aligned}$$

But $\varphi \equiv 1$ and $\gamma \circ P = P$ on A , we have $j^*(\tau)(P) = \omega(P)$. Moreover for any plot $Q: V \rightarrow X$ of X , we have

$$\begin{aligned} d\tau|_{U_A}(Q) &= d\tau(Q|_{Q^{-1}(U_A)}) \\ &= d\omega(\gamma \circ Q|_{Q^{-1}(U_A)}) \\ &= 0. \end{aligned}$$

Similarly, we can prove the condition (1) in the same argument. We shall show the condition (3). Let ω be an element of $\Omega_c^p(X)$ such that $\text{supp}(d\tau) \cap A = \emptyset$ and $j^*(\omega) = 0$. Then $k_2^*(\omega) \in \Omega_c^p(M)$. We have

$$k_1^*[k_2^*(\omega)] = [k_1^*k_2^*(\omega)] = [(k_2 \circ k_1)^*(\omega)] = [j^*(\omega)] = 0.$$

Since $k_1^*: H_{\text{dR}}^p(M) \rightarrow H_{\text{dR}}^p(A) = H_c^p(A)$ is an isomorphism, $[k_2^*(\omega)] = 0$ holds. Thus there exists σ_0 in $\Omega_c^{p-1}(M)$ such that $d\sigma_0 = k_2^*(\omega) = \omega|_M$. Then $\varphi \times \sigma_0$ is in $\Omega_c^{p-1}(M)$ and $d(\varphi \times \sigma_0)|_{U_A} = \omega|_{U_A}$ since $\text{supp} \varphi \subset M$ and $\varphi \equiv 1$ on U_A . Let $\sigma = k_{2*}(\varphi \times \sigma_0) \in \Omega_c^{p-1}(X)$. Clearly, $(d\sigma - \omega)|_{U_A} = 0$ holds. \square

Let $\Omega_c^p(X, A)$ be the kernel of the chain map $j^*: \Omega_c^p(X) \rightarrow \Omega_c^p(A)$. Let us denote the cohomology of the subcomplex $\{\Omega_c^p(X, A), d\}$ by $H_c^p(X, A)$. By the property (1) of Lemma 6.3, we have a short exact sequence:

$$0 \rightarrow \Omega_c^p(X, A) \xrightarrow{l_*} \Omega_c^p(X) \xrightarrow{j^*} \Omega_c^p(A) \rightarrow 0,$$

where l_* is the inclusion map. Thus we have the following long exact sequence:

$$(3) \quad \rightarrow H_c^p(X, A) \xrightarrow{l_*} H_c^p(X) \xrightarrow{j^*} H_c^p(A) \xrightarrow{\delta} H_c^{p+1}(X, A) \rightarrow \dots$$

Now, we get a cochain map $i_*: \Omega_c^p(X \setminus A) \rightarrow \Omega_c^p(X, A)$ since for any τ in $\Omega_c^p(X \setminus A)$, $j^*i_*(\tau) = 0$ holds. Then we have the following.

Proposition 6.4. $i_*: H_c^p(X \setminus A) \rightarrow H_c^p(X, A)$ is an isomorphism.

Proof. We shall show that i_* is injective. Let $[\omega]$ be an element of $\text{Ker } i_*$. Since $i_*[\omega] = 0$ holds, there exists τ in $\Omega_c^{p-1}(X, A)$ such that $d\tau = i_*(\omega)$. Then for any plot P of A , we get $d\tau(P) = i_*(\omega)(P) = 0$. Thus $\text{supp}(d\tau) \cap A$ is empty and $j^*(\tau)$ is zero. Then there exists σ in $\Omega_c^{p-2}(X)$ such that

$$(\tau - d\sigma)|_{U_A} = 0$$

by the property (3) of Lemma 6.3. Hence $\tilde{\tau} = (\tau - d\sigma)|_{X \setminus A}$ is an element of $\Omega_c^{p-1}(X \setminus A)$. Hence we have

$$\begin{aligned} d\tilde{\tau} &= (d\tau - dd\sigma)|_{X \setminus A} \\ &= d\tau|_{X \setminus A} \\ &= i_*(\omega)|_{X \setminus A} \\ &= \omega. \end{aligned}$$

Therefore i_* is injective since $[\omega] = 0$ holds. Next, we shall show that i_* is surjective. Let $[\omega]$ be an element of $H_c^p(X, A)$. Since ω in $Z_c^p(X, A)$, there exists σ in $\Omega_c^{p-1}(X)$ such that $(\omega - d\sigma)|_{U_A} = 0$ by the property (3) of Lemma 6.3. Since we have

$$d(j^*(\sigma)) = j^*(d\sigma) = j^*(\omega) = 0,$$

there exists τ in $\Omega_c^{p-1}(X)$ such that $j^*(\tau) = j^*(\sigma)$ and $d\tau|_{U_A} = 0$ by the property (2) of Lemma 6.3. Then $\sigma - \tau$ is an element of $\Omega_c^{p-1}(X, A)$ since $j^*(\sigma - \tau) = 0$ holds. Let $\tilde{\omega} = (\omega - d(\sigma - \tau))|_{X \setminus A} = (\omega - d\sigma)|_{X \setminus A} + d\tau|_{X \setminus A}$. Then $\tilde{\omega}$ in $\Omega_c^{p-1}(X \setminus A)$ and we get

$$i_*[\tilde{\omega}] = i_*[(\omega - d\sigma)|_{X \setminus A}] = [i_*i^*(\omega - d\sigma)] = [\omega - d\sigma] = [\omega].$$

Therefore i_* is surjective. \square

Therefore we have Theorem 6.2 by the exact sequence (3) and Proposition 6.4.

Corollary 6.5. *Let X be a D -compact diffeological subcartesian space. Let A be a D -closed subset of X . If there exists a D -open subset M of X such that A is a deformation retract of M , then we have a long exact sequence:*

$$\rightarrow H_c^p(X \setminus A) \xrightarrow{i_*} H_{\text{dR}}^p(X) \xrightarrow{j^*} H_{\text{dR}}^p(A) \xrightarrow{\delta} H_c^{p+1}(X \setminus A) \rightarrow \cdots$$

Proof. Clearly, $H_c^p(X) = H_{\text{dR}}^p(X)$ and $H_c^p(A) = H_{\text{dR}}^p(A)$ since A is D -compact by Proposition 2.7 and Proposition 6.1. Therefore the exactness of the sequence above follows from Theorem 6.2. \square

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